Optimization Method

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Example 1

Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. Demand for trains is unlimited, but at most 40 soldiers are bought each week.

A soldier sells for \$27 and uses \$10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by \$14. A train sells for \$21 and uses \$9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by \$10.

The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours.

How to maximize Giapetto's weekly profit?

- **Decision Variables**: The decision variables completely describe the decisions to be made. Denote by x_1 the number of soldiers produced each week, and by x_2 the number of trains produced each week.
- **Objective Function**: The function to be maximized or minimized is called the objective function. Since fixed costs (such as rent and insurance) do not depend on the values of x_1 and x_2, Giapetto can concentrate on maximizing his weekly profit, i.e.,

$$\max \quad 3x_1 + 2x_2.$$

- **Constraints**:
  1. Each week, no more than 100 hours of finishing time may be used.
  2. Each week, no more than 80 hours of carpentry time may be used.
  3. Because of limited demand, at most 40 soldiers should be produced each week.

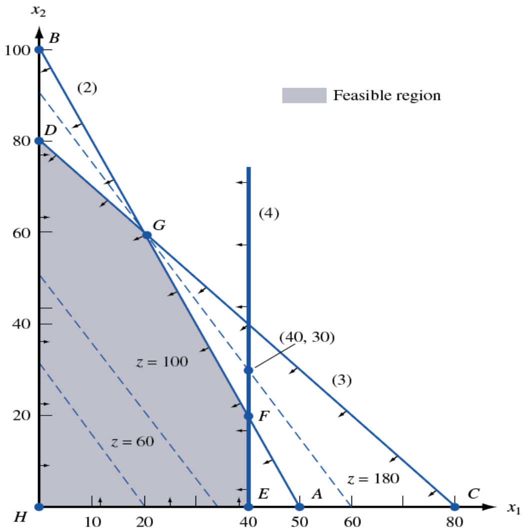
$$2x_1 + x_2 \leq 100, \quad x_1 + x_2 \leq 80, \quad x_1 \leq 40.$$

- **Sign Restrictions**: $x_1 \geq 0$ and $x_2 \geq 0$.

- Linear Programming (LP)
- Integer Programming (IP)
- Dynamic Programming (DP)
- Game Theory

- Simplex Method:
  - The Big M Method
  - The Two-Phase Simplex Method
- Sensitivity Analysis
- Duality

The Graphical Solution



The simplex algorithm proceeds as follows:

Step 1 Convert the LP to standard form.

Step 2 Obtain a bfs (if possible) from the standard form.

Step 3 Determine whether the current bfs is optimal.

Step 4 If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value.

Step 5 Use EROs to find the new bfs with the better objective function value. Go back to step 3.

Example 2

The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given in the following table.

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for \$60, a table for \$30, and a chair for \$20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, Dakota wants to maximize total revenue.

Resource Requirements for Dakota Furniture

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Define the decision variables as x_1 the number of desks produced, x_2 the number of tables produced, and x_3 the number of chairs produced.

$$\begin{aligned} \max \quad & 60x_1 + 30x_2 + 20x_3 \\ \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \quad \text{(Lumber constraint)} \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 \quad \text{(Finishing constraint)} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8 \quad \text{(Carpentry constraint)} \\ & x_2 \le 5 \quad \text{(Limitation on table demand)} \quad x_1, x_2, x_3 \ge 0 \end{aligned}$$

Canonical Form 0

Row		Basic Variable
0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$z = 0$
1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$
2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$
3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$
4	$x_2 + s_4 = 5$	$s_4 = 5$

$$\text{BV} = \{z, s_1, s_2, s_3, s_4\} \text{ and NBV} = \{x_1, x_2, x_3\}.$$

Is the Current Basic Feasible Solution Optimal?

If we solve for z by rearranging row 0, then we obtain

$$z = 60x_1 + 30x_2 + 20x_3.$$

Because a unit increase in x_1 causes the largest rate of increase in z, we choose to increase x_1 from its current value of zero (the **entering variable**). Observe that x_1 has the most negative coefficient in row 0.

How large we can make x_1?

$$s_1 \geq 0 \quad \text{for} \quad x_1 \leq \frac{48}{8} = 6, \quad s_2 \geq 0 \quad \text{for} \quad x_1 \leq \frac{20}{4} = 5,$$
$$s_3 \geq 0 \quad \text{for} \quad x_1 \leq \frac{8}{2} = 4, \quad s_4 \geq 0 \quad \text{for all values of } x_1.$$

Therefore, we have

$$x_1 = \min\{6, 5, 4\} = 4.$$

Definition 3.1 (The Ratio Test)

When entering a variable into the basis, compute the ratio for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called the **winner of the ratio test**. *The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative.*

To make x_1 a basic variable in row 3, we use EROs to make x_1 have a coefficient of 1 in row 3 and a coefficient of 0 in all other rows. The final result is that x_1 replaces s_3 as the basic variable for row 3 (**pivoting, pivot row, pivot term**). Now BV $= \{z, s_1, s_2, x_1, s_4\}$ and NBV $= \{s_3, x_2, x_3\}$.

Canonical Form 1

Row		Basic Variable
Row 0′	$z + 15x_2 - 5x_3 + 30s_3 = 240$	$z = 240$
Row 1′	$- x_3 + s_1 - 4s_3 = 16$	$s_1 = 16$
Row 2′	$- x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	$s_2 = 4$
Row 3′	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	$x_1 = 4$
Row 4′	$x_2 + s_4 = 5$	$s_4 = 5$

Definition 3.2

The procedure used to go from one bfs to a better adjacent bfs is called an **iteration** *(or sometimes, a* **pivot***) of the simplex algorithm.*

We now try to find a bfs that has a still larger z-value by examining canonical form 1,

$$z = 240 - 15x_2 + 5x_3 - 30s_3.$$

After one more iteration, we reach

Canonical Form 2

Row		Basic Variable
0″	$z + 5x_2 + 10s_2 + 10s_3 = 280$	$z = 280$
1″	$-2x_2 + s_1 + 2s_2 - 8s_3 = 24$	$s_1 = 24$
2″	$-2x_2 + x_3 + 2s_2 - 4s_3 = 8$	$x_3 = 8$
3″	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$	$x_1 = 2$
4″	$x_2 + s_4 = 5$	$s_4 = 5$

$$\text{BV} = \{z, s_1, x_3, x_1, s_4\} \quad \text{and} \quad \text{NBV} = \{s_2, s_3, x_2\}.$$

If we rearrange row $0''$ and solve for z, we obtain

$$z = 280 - 5x_2 - 10s_2 - 10s_3.$$

Our current bfs from canonical form 2 is an optimal solution.

Step 1 Convert the LP to standard form.

Step 2 Find a bfs. This is easy if all the constraints are $\leq$ with nonnegative right-hand sides as the slack variable s_i may be used as the basic variable for row i. If no bfs is readily apparent, then use the techniques to be discussed in later sections.

Step 3 If all nonbasic variables have **nonnegative** coefficients in row 0, then the current bfs is **optimal**. If any variables in row 0 have negative coefficients, then choose the variable with the most negative coefficient in row 0 to enter the basis.

Step 4 Use EROs to make the entering variable the basic variable in any row that wins the ratio test (ties may be broken arbitrarily). After the EROs have been used to create a new canonical form, return to step 3, using the current canonical form.

A Graphical Introduction to Sensitivity Analysis

Definition 3.3

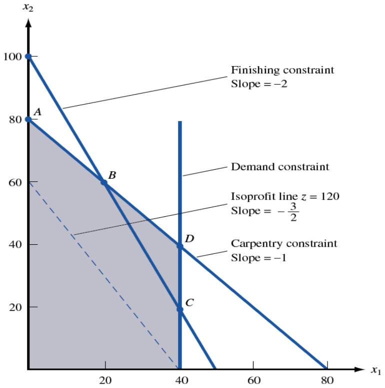
Sensitivity analysis *is concerned with how changes in an LP's parameters affect the optimal solution.*

Reconsider the Giapetto problem:

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \le 100, \\ & x_1 + x_2 \le 80, \\ & x_1 \le 40, \\ & x_1, x_2 \ge 0. \end{aligned}$$

The optimal solution is $z = 180$, $x_1 = 20$, $x_2 = 60$. How would changes in the problem's objective function coefficients or right-hand sides change this optimal solution?

Let c_1 be the contribution to profit by each soldier. For what values of c_1 does the current basis remain optimal?



If a change in c_1 causes the isoprofit lines to be flatter than the carpentry constraint, then the optimal solution will change from the current optimal solution (point B) to a new optimal solution (point A).

Definition 3.4

Associated with any LP is another LP, called the **dual**. *When taking the dual of a given LP, we refer to the given LP as the* **primal**.

A **normal** max **problem** may be written as

$$\max \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{s.t.} \quad \begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\ \ldots & & \ldots & & & & \ldots & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leq & b_m \end{array}, \quad x_j \geq 0 \tag{1}$$

The dual of a normal max problem (**normal** min **problem**) is defined to be

$$\min \quad b_1y_1 + b_2y_2 + \ldots + b_my_m$$

$$\text{s.t.} \quad \begin{array}{ccccccccc} a_{11}y_1 & + & a_{21}y_2 & + & \ldots & + & a_{m1}y_m & \geq & c_1 \\ a_{12}y_1 & + & a_{22}y_2 & + & \ldots & + & a_{m2}y_m & \geq & c_2 \\ \ldots & & \ldots & & & & \ldots & & \\ a_{1n}y_1 & + & a_{2n}y_2 & + & \ldots & + & a_{mn}y_m & \geq & c_n \end{array}, \quad y_i \geq 0 \tag{2}$$

The Dual Simplex Method (max)

Suppose the initial simplex tableau is **dual feasible**.

Step 1 Is the right-hand side of each constraint nonnegative? If so, an optimal solution has been found; if not, at least one constraint has a negative right-hand side, and we go to step 2.

Step 2 Choose the most negative basic variable as the variable to leave the basis. The row in which the variable is basic will be the pivot row. To select the variable that enters the basis, we compute the following ratio for each variable x_j that has a negative coefficient in the pivot row, and choose the one with the smallest ratio (absolute value):

$$\frac{\text{Coefficient of } x_j \text{ in row } 0}{\text{Coefficient of } x_j \text{ in pivot row}}.$$

Step 3 If there is any constraint in which the right-hand side is negative and each variable has a nonnegative coefficient, then the LP has no feasible solution. If no constraint indicating infeasibility is found, return to step 1.

Changing a Right-Hand Side

If the rhs of a constraint is changed and the current basis becomes infeasible, the dual simplex can be used to find the new optimal solution.

Suppose that 30 finishing hours are now available.

"Old" Optimal Dakota Tableau If 30 Finishing Hours Are Available

	Basic Variable
$z + 5x_2 + 10s_2 + 10s_3 = 380$	$z = 380$
$- 2x_2 + s_1 + 2s_2 - 8s_3 = 44$	$s_1 = 44$
$- 2x_2 + x_3 + 2s_2 - 4s_3 = 28$	$x_3 = 28$
$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = -3$	$x_1 = -3$

"New" Optimal Dakota Tableau If 30 Finishing Hours Are Available

	Basic Variable
$z + 20x_1 + 30x_2 + 40s_3 = 320$	$z = 320$
$4x_1 + 3x_2 + s_1 - 2s_3 = 32$	$s_1 = 32$
$4x_1 + 3x_2 + x_3 + 2s_3 = 16$	$x_3 = 16$
$- 2x_1 - 2.5x_2 + s_2 - 3s_3 = 6$	$x_1 = 6$

- Matlab
- Cplex
- Gurobi
- Lingo or Lindo
- ...

Definition 3.5

An **integer programming problem** *(IP) is an LP in which some or all of the variables are required to be non-negative integers, i.e., the Divisibility Assumption in LP does not hold.*

A **pure integer programming problem**

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 6 \quad x_1, x_2 \ge 0, \ x_1, x_2 \text{ integer.} \end{aligned}$$

A **mixed integer programming problem**

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 6, \quad x_1, x_2 \ge 0, \ x_1 \text{ integer.} \end{aligned}$$

A **0-1 integer programming problem**

$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 2, \quad 2x_1 - x_2 \le 1, \quad x_1, x_2 = 0 \text{ or } 1. \end{aligned}$$

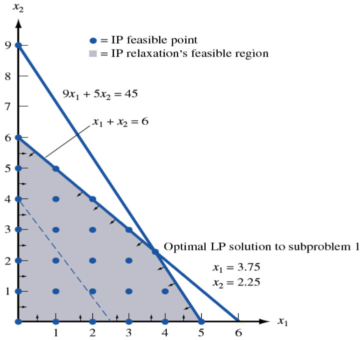
If you solve the LP relaxation of a pure IP and obtain a solution in which all variables are integers, then the optimal solution to the LP relaxation is also the optimal solution to the IP.

Example 3

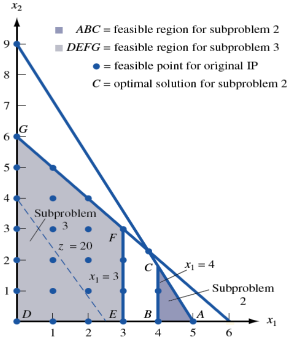
The Telfa Corporation manufactures tables and chairs. A table requires 1 hour of labor and 9 square board feet of wood, and a chair requires 1 hour of labor and 5 square board feet of wood. Currently, 6 hours of labor and 45 square board feet of wood are available. Each table contributes $8 to profit, and each chair contributes $5 to profit.

Formulate and solve an IP to maximize Telfa's profit.

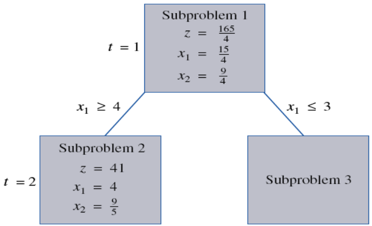
$$\begin{aligned} \max \quad & 8x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 6 \quad \text{(Labor constraint)} \\ & 9x_1 + 5x_2 \leq 45 \quad \text{(Wood constraint)}, \quad x_1, x_2 \geq 0,\ x_1, x_2 \text{ integer.} \end{aligned}$$



x_2
● = IP feasible point
■ = IP relaxation's feasible region
$9x_1 + 5x_2 = 45$
$x_1 + x_2 = 6$
Optimal LP solution to subproblem 1
$x_1 = 3.75$
$x_2 = 2.25$
x_1



x_2
ABC = feasible region for subproblem 2
DEFG = feasible region for subproblem 3
● = feasible point for original IP
C = optimal solution for subproblem 2
Subproblem 3
$z = 20$
$x_1 = 3$
$x_1 = 4$
Subproblem 2
x_1



Subproblem 1
$z = \frac{165}{4}$
$x_1 = \frac{15}{4}$
$x_2 = \frac{9}{4}$
$t = 1$
$x_1 \geq 4$
$x_1 \leq 3$
Subproblem 2
$z = 41$
$x_1 = 4$
$x_2 = \frac{9}{5}$
$t = 2$
Subproblem 3

The key aspects of the branch-and-bound method for solving pure IPs may be summarized as follows:

Step 1 If it is unnecessary to branch on a subproblem, then it is fathomed. The following three situations result in a subproblem being fathomed:

(1) The subproblem is infeasible;
(2) the subproblem yields an optimal solution in which all variables have integer values; and
(3) the optimal z-value for the subproblem does not exceed (in a max problem) the current LB.

Step 2 A subproblem may be eliminated from consideration in the following situations:

(1) The subproblem is infeasible (in the Telfa problem, subproblem 4 was eliminated for this reason);
(2) the LB (representing the z-value of the best candidate to date) is at least as large as the z-value for the subproblem (in the Telfa problem, subproblems 3 and 7 were eliminated for this reason).

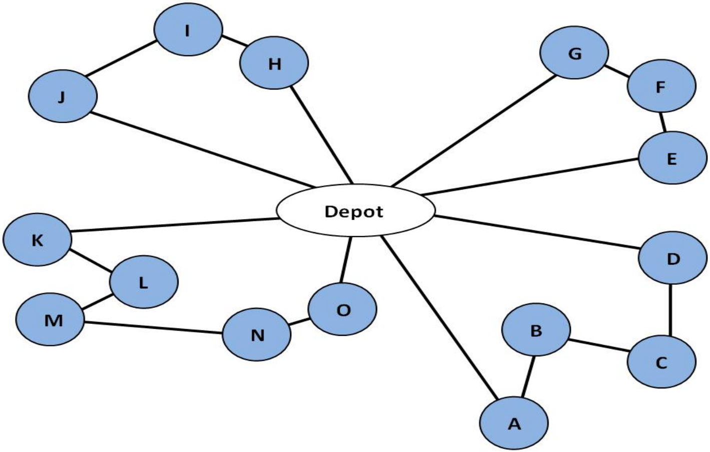


Figure: Technician Routing Problem with Experience-based Service Times

Technician Routing Problem with Experience-based Service Times (TRP-EST)

- Technicians have varying levels of experience across the tasks.
- The experience depends on the number of times the technician has performed the task.
- The more experienced a technician is, the less time he/she needs to complete the task.
- The objective is to minimize the expected sum of the completion time of the last task for each day.

Modeling the Problem as a Markov Decision Process

- Decision epochs correspond to days
- States correspond to experience levels of technicians on tasks
- Actions correspond to a feasible set of routes that satisfy following constraints:

$$\sum_{i\in\mathcal{C}_t\cup\{0\}} \sum_{k\in\mathcal{K}} x_{ijt}^k = 1 \qquad \forall j \in \mathcal{C}_t, \tag{3}$$

$$\sum_{j\in\mathcal{C}_t\cup\{C_t+1\}} x_{0jt}^k = 1 \qquad \forall k \in \mathcal{K}, \tag{4}$$

$$\sum_{i\in\mathcal{C}_t\cup\{0\}} x_{i(C_t+1)t}^k = 1 \qquad \forall k \in \mathcal{K}, \tag{5}$$

$$\sum_{j\in\mathcal{C}_t\cup\{0\}} x_{jit}^k - \sum_{j\in\mathcal{C}_t\cup\{C_t+1\}} x_{ijt}^k = 0 \qquad \forall i \in \mathcal{C}_t, \forall k \in \mathcal{K}, \tag{6}$$

$$B_j \geq B_i + \sum_{k\in\mathcal{K}}\left(\sum_{r\in\mathcal{R}} z_{ir} d_{it}^k + \tau_{ij}\right)x_{ijt}^k \qquad \forall i \in \mathcal{C}_t \cup \{0\}, \forall j \in \mathcal{C}_t, \tag{7}$$

$$x_{it}^k \in \{0,1\} \qquad \forall i \in \mathcal{C}_t, \forall k \in \mathcal{K}, \tag{8}$$

$$B_i \geq 0 \qquad \forall i \in \mathcal{C} \cup \{0, C_t+1\}. \tag{9}$$

Modeling the Problem as a Markov Decision Process

- Exogenous information corresponds to today's customer requests.
- Transition function: $q^k_{r,t+1}(s_t, a_t) = q^k_{rt} + \sum_{i \in \mathcal{C}_t : r(i)=r} x^k_{it}$
- Contribution function: $c(s_t, a_t) = e^t_{max} \quad \forall a_t \in A_t(s_t)$
- Objective function: $\min_{\pi \in \Pi} E\left[\sum_{t=1}^{T} C(s_t, \delta^\pi_t(s_t))\right]$
- Bellman's equation:

$$V(s_t) = \min_{a_t \in \mathcal{A}_t(s_t)} \{c(s_t, a_t) + E\left[V(s_{t+1}) \mid s_t, a_t\right]\}.$$

Solution Approach: Approximate Dynamic Programming

Approximate the value functions:

- Lookup tables
- Parametric models
- Nonparametric models

Stackelberg Leadership Model

- Players: Leader and follower.
- Objective: Find the subgame perfect Nash equilibrium or equilibria (SPNE)
- Backward induction:
  1. The leader considers what the best response of the follower is, i.e. how it will respond once it has observed the quantity of the leader.
  2. The leader then picks a quantity that maximises its payoff, anticipating the predicted response of the follower.
  3. The follower actually observes this and in equilibrium picks the expected quantity as a response.

Stackelberg Leadership Model

The profit of firm 2 (the follower) is revenue minus cost:

$$\Pi_2 = P(q_1 + q_2) \cdot q_2 - C_2(q_2) \tag{10}$$

To maximize Π_2, set

$$\frac{\partial \Pi_2}{\partial q_2} = 0 \tag{11}$$

Stackelberg leadership model

The profit of firm 1 (the leader) is revenue minus cost:

$$\Pi_1 = P(q_1 + q_2) \cdot q_1 - C_1(q_1) \tag{12}$$

To maximize Π_1, set

$$\frac{\partial \Pi_1}{\partial q_1} = 0 \tag{13}$$

- LaTeX: A document preparation system
- TeXShop: For IOS
- CTEX: TeX with Chinese